The supremum in (2.9) is finite if

$$\|\partial_t V_t\ U(t,0)\ \Psi\| \le \|\partial_t V_t\|\ \|U(t,0)\ \Psi - \mathrm{e}^{-iH_0t}\ \Phi\| + \|\partial_t V_t\ \mathrm{e}^{-iH_0t}\ \Phi\|$$

is integrable on $\pm\, t \in [0,\infty)$. By Assumption (2.5) this follows for the second term on the r.h.s. for a total set of states Ψ.

For $\Psi = \Omega^{\pm}\ \Phi$ we have $\lim_{t\to+\infty} U(t,0)^*\ \mathrm{e}^{-iH_0t}\ \Phi = \Psi$. Thus

$$\begin{aligned}\|U(t,0)\ \Psi - \mathrm{e}^{-iH_0t}\ \Phi\| &= \|\Psi - U(t,0)^*\ \mathrm{e}^{-iH_0t}\ \Phi\| \\ &\le \int_s^{\infty} \mathrm{d}s\, \|V_s\ \mathrm{e}^{-iH_0s}\ \Phi\| = \int_t^{\infty} \mathrm{d}s\, \frac{h(s)}{1+|s|}\end{aligned}$$

for some integrable function $h \in L^1([0,+\infty))$ by Assumption (2.4). Using partial integration we conclude integrability:

$$\begin{aligned}\int_0^{\infty} \mathrm{d}t \int_t^{\infty} \mathrm{d}s\, \frac{h(s)}{1+s} &= t \int_t^{\infty} \mathrm{d}s\, \frac{h(s)}{1+s}\Big|_{t=0}^{t=\infty} + \int_0^{\infty} \mathrm{d}t\, \frac{t}{1+t} h(t) \\ &\le \int_0^{\infty} \mathrm{d}s\, h(s) < \infty.\end{aligned}$$

Consequently, the time derivative (2.10) is integrable on $[0,\infty)$ and the supremum (2.9) is finite for a total set of $\Psi = \Omega^{+}\ \Phi$, $t \ge 0$. The uniform boundedness for $t \le 0$ and $\Psi = \Omega^{-}\ \Phi$ is proved similarly. □

Next we will give sufficient conditions which guarantee that (2.4) and (2.5) are satisfied. For simplicity of presentation we use standard nonrelativistic kinematics (1.1), $H_0 = p^2/2m$. We will apply geometrical time-dependent methods. Then a convenient total set $\mathcal{D}_0 \subset \mathcal{H}$ consists of states with good localization in momentum space. Let $\widehat{\varphi}(\mathbf{p})$ denote the momentum space wave function of Φ and $B_{mv/3}(m\mathbf{v}) \subset \mathbb{R}^{\nu}$ the open ball of radius $mv/3$ with center $m\mathbf{v} \in \mathbb{R}^{\nu}$, $\mathbf{v} \ne 0$, $v = |\mathbf{v}|$. We choose the set $\mathcal{D}_0$ as

$$\begin{aligned}\mathcal{D}_0\ :=\ \{\Phi \in \mathcal{H} \mid \|\Phi\| = 1,\ \widehat{\varphi} \in C_0^{\infty}(\mathbb{R}^{\nu}),\ \exists \mathbf{v} \in \mathbb{R}^{\nu}, \mathbf{v} \ne 0, \\ \text{such that } \operatorname{supp} \widehat{\varphi} \subseteq B_{mv/3}(m\mathbf{v})\}. \qquad (2.11)\end{aligned}$$

Any state Ψ with $\widehat{\psi} \in C_0^{\infty}(\mathbb{R}^{\nu})$, $0 \notin \operatorname{supp} \widehat{\psi}$ can be written as a finite linear combination of vectors in $\mathcal{D}_0$. This set is dense in $L^2(\mathbb{R}^{\nu}) = \mathcal{H}$.

The states in $\mathcal{D}_0$ propagate mainly into regions where $\mathbf{x} \approx t\mathbf{p}/m \approx t\mathbf{v}, \mathbf{p} \in \operatorname{supp} \widehat{\varphi}$. More precisely, one shows with a stationary phase estimate that propagation into 'classically forbidden' regions decays rapidly:

$$\|F(|\mathbf{x} - t\mathbf{v}| \ge \rho + |t|v/2)\, \mathrm{e}^{-itH_0}\Phi\| \le C_N(1+\rho+|t|)^{-N},\ N \in \mathbb{N}, \rho \ge 0, \qquad (2.12)$$

with a constant $C_N = C_N(\Phi) < \infty$ (see, e.g., §II of [4]). Similar estimates hold for other kinematics. We will use this bound for $\rho = 0$ here and with $\rho > 0$ in the last section.

While the estimate (2.12) follows from propagation of wave packets one has, in addition, the standard estimate of spreading in $\mathbb{R}^{\nu}$,

$$\sup_{x \in \mathbb{R}^{\nu}} |(\mathrm{e}^{-iH_0t}\ \Phi)(x)| \quad \le \quad C(\Phi)\ (1+|t|)^{-\nu/2}, \qquad (2.13)$$

where $C(\Phi) < \infty$ for $\Phi \in \mathcal{D}_0$.

Now we return to the rotating potentials (1.2) which are possible in $\nu \geq 2$ dimensions. We will give sufficient conditions for the two dimensional case which is the 'worst case': the falloff (2.13) is slowest and – compared to $\mathbb{R}^3$ – the potential does not decay in the direction parallel to the axis of rotation. We may use polar coordinates (r, ϕ) in the (x_1, x_2)-plane.

The potential can be decomposed into a rotationally invariant part

$$V_{\text{inv}}(x) \quad := \quad \frac{\omega}{2\pi} \int_0^{2\pi/\omega} V(\mathcal{R}(t)^{-1} x) \, \mathrm{d}t$$

and the rest $V_{\text{noninv}} = V - V_{\text{inv}}$. The rotationally invariant part of the potential remains time-independent. It need not be bounded nor differentiable and it does not show up in (2.5). If for every $g \in C_0^\infty(\mathbb{R})$ there is an integrable $\widetilde{h} \in L^1([0, \infty))$ (e.g., $\widetilde{h}(\rho) = C(1 + \rho)^{-1-\varepsilon}$) such that

$$\|V_{\text{inv}} \, g(H_0) \, F(|x| > \rho)\| \quad \leq \quad \frac{\widetilde{h}(\rho)}{1 + \rho}, \tag{2.14}$$

then (2.4) is satisfied for V_{inv}: For $\Phi \in \mathcal{D}_0$ choose $g \in C_0^\infty(\mathbb{R})$ such that $g(H_0)\Phi = \Phi$. Then

$$\begin{aligned} \|V_{\text{inv}} \, \mathrm{e}^{-iH_0 t} \, g(H_0) \, \Phi\| &\leq \|V_{\text{inv}} \, g(H_0) \, F(|x| > |t|v/2)\| \, \|\Phi\| \\ &\quad + \|V_{\text{inv}} \, g(H_0)\| \, \|F(|x| < |t|v/2) \, \mathrm{e}^{-iH_0 t} \Phi\| \\ &\leq \frac{\widetilde{h}(|t|v/2)}{1 + |t|v/2} + O(|t|^{-N}) = \frac{h(t)}{1 + |t|} \end{aligned}$$

with $h \in L^1$ by (2.14) and (2.12).

Lemma 2.2. *Let V be Kato-bounded and let there exist an integrable function $h \in L^1([0, \infty))$ such that the potential V satisfies the condition*

$$\rho \, \|V \; F(|\mathbf{x}| > \rho)\| \leq h(\rho) \tag{2.15}$$

or one of the weaker conditions

$$\rho \, \|V \; (H_0 + 1)^{-1} \; F(|\mathbf{x}| > \rho)\| \leq h(\rho) \tag{2.16}$$

or for every $g \in C_0^\infty(\mathbb{R})$ there is an integrable $h = h_g$ with

$$\rho \, \|V \; g(H_0) \; F(|\mathbf{x}| > \rho)\| \leq h(\rho). \tag{2.17}$$

Then the rotating potential $V_t = V(\mathcal{R}(t)^{-1} \cdot)$ satisfies (2.4), *i.e., for every $\Phi \in \mathcal{D}_0$* (2.11) *there is an integrable $\widetilde{h}$ such that*

$$|t| \, \|V_t \; \mathrm{e}^{-itH_0} \; \Phi\| \leq \widetilde{h}(|t|).$$

If the partial (*distributional*) *azimuthal derivative $(\partial_\phi V)(r, \phi)$ yields a bounded multiplication operator $\partial_\phi V$ which satisfies*

$$\|\partial_\phi V \; F(|\mathbf{x}| > \rho)\| \leq h(\rho) \tag{2.18}$$

or the weaker

$$\|\partial_\phi V\ (H_0+1)^{-1}\ F(|\mathbf{x}|>\rho)\| \le h(\rho) \tag{2.19}$$

or for every $g \in C_0^\infty(\mathbb{R})$

$$\|\partial_\phi V\ g(H_0)\ F(|\mathbf{x}|>\rho)\| \le h(\rho) \tag{2.20}$$

for some integrable h then (2.5) *holds, i.e., for every $\Phi \in \mathcal{D}_0$ there is an integrable $\widetilde{h}$ with*

$$\|\partial_t V_t\ \mathrm{e}^{-itH_0}\ \Phi\| \le \widetilde{h}(|t|).$$

Remarks. If (2.15) holds then it implies (2.16) and (2.17) because the regularizing factors $(H_0+1)^{-1}$ or $g(H_0)$ act in configuration space as convolutions with a continuous rapidly decreasing function. Thus the required decay rate is preserved. But even if the operators on the l.h.s. of (2.15) are bounded the decay rate may be better in the regularized versions (2.16) or (2.17): think of a sequence of 'dipole' pairs of peaks with maxima and minima of equal amplitude but 'closer and thinner' pairs when they are localized farther away. Then $\|V\ \ F(|\mathbf{x}|>\rho)\|$ does not decay but the convolution causes falloff due to cancellations. The same applies to conditions (2.18)–(2.20).

A potential $V(r,\phi)$ which in an angular sector behaves like

$$V(r,\phi) = \frac{1}{r^2\ (\ln r)^2}\ \cos(r^\alpha\ \phi),\ \ r>2,\ \ \phi_1<\phi<\phi_2,$$

satisfies in this region (2.15) and (2.18) for exponents $0 \le \alpha \le 1$ but the latter is violated for $\alpha > 1$. A behavior like $\alpha = 1$ will show up in the next example.

Proof of Lemma 2.2. Since $\Phi \in \mathcal{D}_0$ has compact support in momentum space we may choose $g \in C_0^\infty(\mathbb{R})$ such that $g(H_0)\ \Phi = \Phi$. Due to rotational invariance of H_0 and $|\mathbf{x}|$ we have

$$\begin{aligned}\|V_t\ g(H_0)\ F(|\mathbf{x}|>\rho)\| &= \|V\ g(H_0)\ F(|\mathbf{x}|>\rho)\|,\\ \|\partial_t V_t\ g(H_0)\ F(|\mathbf{x}|>\rho)\| &= \omega\ \|\partial_\phi V\ g(H_0)\ F(|\mathbf{x}|>\rho)\|.\end{aligned}$$

To estimate (2.4) we use (2.17) and (2.12):

$$\begin{aligned}\|V_t\ \mathrm{e}^{-itH_0}\ \Phi\| &\le \|V\ g(H_0)\ F(|\mathbf{x}|>|t|\,v/2)\|\ \|\Phi\|\\ &\quad + \|V\ g(H_0)\|\ \|F(|\mathbf{x}|<|t|\,v/2)\ \mathrm{e}^{-itH_0}\ \Phi\|\\ &\le \frac{1}{1+|t|\,v/2}\,h(|t|\,v/2) + O(|t|^{-N}).\end{aligned}$$

Similarly, (2.20) and (2.12) yield (2.5).

In the case of regularization with a resolvent observe that $(H_0+1)^{-1}\ \ \Phi/\|(H_0+1)^{-1}\ \ \Phi\| \in \mathcal{D}_0$ has the same smoothness and support properties in momentum space as Φ. □

Another geometrical configuration is described by a strongly anisotropic potential localized near a hyperplane, in $\nu = 2$ dimensions near a line. For simplicity we assume that the support is bounded in the x_2-direction, a sufficiently rapid decay would give the same result. Moreover, we state the lemma for differentiable potentials in product form, the generalization to less regular ones as in the previous lemma is straightforward.

Lemma 2.3. *Let the potential* $V(x_1, x_2) = V^{(1)}(x_1)\, V^{(2)}(x_2) \in C^1(\mathbb{R}^2)$ *satisfy* $\operatorname{supp} V^{(2)} \subset [-d, d]$ *and the bound*

$$\rho^{1/2} \sup_{|x_1| \geq \rho} \left| V^{(1)}(x_1) \right| + \left(\frac{1}{1+\rho} \right)^{1/2} \sup_{|x_1| \geq \rho} \left| \frac{\mathrm{d}}{\mathrm{d}x_1} V^{(1)}(x_1) \right| \leq h(\rho) \tag{2.21}$$

for some integrable h. *Then* $V_t = V(\mathcal{R}(t)^{-1} \cdot)$ *satisfies conditions* (2.4) *and* (2.5) *for every* $\Phi \in \mathcal{D}_0$.

Proof. Up to rapidly decaying parts which do not affect the integrability the configuration space wave function is localized in a moving disk and satisfies for large $|t|$ the estimate

$$\left| \left(\mathrm{e}^{-itH_0}\, \Phi \right)(\mathbf{x}) \right| \leq \frac{\text{const}}{|t|}\, \chi_{B_{|t|\,v/2}(t\mathbf{v})}(\mathbf{x})$$

by (2.12) and (2.13). $\chi_{B_{|t|\,v/2}(t\mathbf{v})}$ denotes the characteristic function of $B_{|t|\,v/2}(t\mathbf{v})$. The k-th passage of a 'tail' of the rotating potential takes place around $t_k = k\pi/\omega$ and lasts less than $2\tau = \pi/\omega$ (for $|t| > 5d/v$). The area of intersection of the disk with the support of the potential is bounded by $\mathrm{d}v\,(|t_k| + \tau)$ and

$$|V(\mathbf{x})| \leq \sup\, |V^{(2)}|\ \frac{1}{v\,(|t_k| - \tau)/2}\, h(v\,(|t_k| - \tau)/2),$$
$$\mathbf{x} \in B_{|t|\,v/2}(t\mathbf{v}),\ |t| \geq |t_k| - \tau.$$

For given $\mathbf{v}$ and ω we obtain for one passage (up to rapidly decaying terms)

$$\begin{aligned} &\int_{t_k-\tau}^{t_k+\tau} \mathrm{d}t\ \left\| V_t\, \mathrm{e}^{-itH_0}\, \Phi \right\| \\ &\leq 2\tau\ \frac{1}{v\,(|t_k| + \tau)/2}\, h(v\,(|t_k| + \tau)/2)\ \ \frac{\text{const}}{|t_k| - \tau}\ \{\mathrm{d}v(|t_k| + \tau)\}^{1/2} \\ &\leq \frac{\text{const}}{|t_k| + \tau}\ h(\text{const}\,|t_k|) \end{aligned} \tag{2.22}$$

for large enough $|k|$. Since $\|V_t\, \mathrm{e}^{-itH_0}\, \Phi\|$ is bounded on compact intervals the estimate (2.22) shows (2.4).

With $\partial_t V_t = \omega[\, x_2 \partial_1 V - x_1 \partial_2 V\,](\mathcal{R}(t)^{-1} \cdot)$ the first summand yields a bound on $B_{|t|\,v/2}(t\mathbf{v})$, $|t| \geq |t_k| - \tau$,

$$\omega\ \sup_{x_2} \left| x_2\, V^{(2)}(x_2) \right|\ [\, v\,(|t_k| - \tau)/2\,]^{1/2}\, h(v\,(|t_k| - \tau)/2)$$

by (2.21) while the second is bounded there by

$$\omega\ \sup_{x_2} \left| \frac{\mathrm{d}}{\mathrm{d}x_2}\, V^{(2)}(x_2) \right|\ \frac{3v\,(|t_k| - \tau)/2}{[v\,(|t_k| - \tau)/2]^{1/2}}\ h(v\,(|t_k| - \tau)/2).$$

Combining these estimates as above shows (2.5). □

Our third example demonstrates how dimensions strictly larger than two help if the potential decays in the other directions. For simplicity we assume $\nu = 3$ and compact support in the vertical direction (parallel to the axis of rotation) of a differentiable potential. Note that we do not need any falloff in the plane of rotation to show boundedness of the kinetic energy for asymptotically free scattering states. (The existence of wave operators follows easily for such potentials but one will need additional assumptions for asymptotic completeness.)

Lemma 2.4. *Let* $V \in C^1(\mathbb{R}^3)$ *have bounded* C^1*-norm and satisfy* $\operatorname{supp} V \subset \{\mathbf{x} \in \mathbb{R}^3 \mid |x_3| \leq d\}$. *Then* (2.4) *and* (2.5) *are satisfied.*

Proof. Let $\mathcal{D}_0$ be the total set of states with $\widehat{\varphi} \in C_0^\infty(\mathbb{R}^3)$, for which there exists a constant $b > 0$ such that either $\operatorname{supp} \widehat{\varphi} \subset \{\mathbf{p} \in \mathbb{R}^3 \mid p_3 > mb\}$ or $\operatorname{supp} \widehat{\varphi} \subset \{\mathbf{p} \in \mathbb{R}^3 \mid p_3 < -mb\}$. Then $\|F(|x_3| < |t|b/2)\ \mathrm{e}^{-itH_0}\ \Phi\| = O(|t|^{-N})$ and conditions (2.4) and (2.5) follow. □

To sum up the results of this section: If one knows (using any method) unitarity of the scattering operator or even asymptotic completeness and if the potential can be split into a sum of terms which satisfy any of the above sufficient conditions, then the kinetic energy is bounded uniformly in time in both time-directions simultaneously on the corresponding subspace of asymptotically free scattering states.

3. Evolution in a rotating frame

Here we study the time evolution in a rotating frame for potentials which no longer have to be smooth. This transformation yields an explicit formula for the propagator $U(t, s)$ in terms of the unitary group for some time-independent generator. This will allow to apply methods of stationary scattering theory to show existence and completeness of the wave operators in § 4.

Let $\mathcal{R}(t) \mapsto R(t)$ be the standard unitary representation of the one-parameter group $\mathcal{R}(t)$ in $L^2(\mathbb{R}^\nu)$, i.e., $(R(t)\psi)(x) = \psi(\mathcal{R}(t)^{-1}x)$. Let ωJ denote its generator, $R(t) = \exp\{-i\omega t J\}$. On a suitable domain the operator J is of the form $x_1(-i\partial/\partial x_2) - x_2(-i\partial/\partial x_1)$ or $-i\partial/\partial\phi$ if one uses cartesian or polar coordinates, respectively, in the x_1, x_2-plane.

For an observer in a rotating reference frame which turns around the orgin like the potential the latter becomes time-independent

$$V_t = R(t)\ V\ R(t)^* \longrightarrow R(t)^*\ V_t\ R(t) = V.$$

Let $t \mapsto \Psi(t) = U_{\mathrm{inert}}(t, s)\,\Psi(s)$ be *any* time evolution in the given inertial frame with propagator U_{inert}. Then an observer in the rotating frame will see

$$R(t)^*\,\Psi(t) = R(t)^*\,U_{\mathrm{inert}}(t, s)\,\Psi(s) = R(t)^*\,U_{\mathrm{inert}}(t, s)\,R(s)\ R(s)^*\,\Psi(s)$$

with propagator

$$U_{\mathrm{rot}}(t, s) = R(t)^*\,U_{\mathrm{inert}}(t, s)\,R(s). \tag{3.1}$$

The free time evolution of a state then becomes

$$R(t)^*\ \mathrm{e}^{-itH_0}\ \Psi = \mathrm{e}^{it\omega J}\ \mathrm{e}^{-itH_0}\ \Psi, \tag{3.2}$$

where $\mathrm{e}^{-itH_0}\,\Psi$ is the free time evolution in the inertial frame generated by H_0 as in (1.1) (or any other spherical free Hamiltonian like the relativistic one). Time zero (or $k\,2\pi/\omega$, $k \in \mathbb{Z}$) is singled out by the fact that the rotating and inertial frames coincide and the fixed potential $V_t|_{t=0} = V$ has been picked out of the family V_t for this reference time. Although the free time evolution is rotation invariant we have a different 'unperturbed' evolution which combines the unchanged free evolution with the rotation. Instead of a motion with constant velocity the unperturbed motion now is along spirals.

As the groups in (3.2) commute their product is again a unitary group with a self-adjoint generator denoted by H_ω

$$\mathrm{e}^{i\omega J}\,\mathrm{e}^{-itH_0} =: \mathrm{e}^{-itH_\omega}.$$

Formally we have

$$H_\omega = H_0 - \omega J \tag{3.3}$$

but the domains differ. All three operators are essentially self-adjoint on each of the sets

$$\mathcal{D} := \{\Psi \in \mathcal{H} \mid \widehat{\psi} \in C_0^\infty(\mathbb{R}^\nu)\} \subset \mathcal{S}(\mathbb{R}^\nu) \subset \mathcal{D}(H_0) \cap \mathcal{D}(J), \tag{3.4}$$

where $\mathcal{D}$ is the set of states with smooth compactly supported wave functions in momentum space, $\mathcal{S}(\mathbb{R}^\nu)$ the Schwartz space of smooth rapidly decreasing functions (in configuration or momentum space) and $\mathcal{D}(A)$ denotes the domain of a self-adjoint operator A. All these sets are cores because they are dense in $L^2(\mathbb{R}^\nu)$ and invariant under each of the groups (see, e.g., ([13], Theorem VIII. 11)).

The operator (3.3) has been previously studied by Tip [15] in connection with the circular AC Stark effect. Let P_j, $j \in \mathbb{Z}$ denote the projection onto the eigenspace of J. Since H_0 and J commute, the subspaces $\mathcal{H}_j = P_j\mathcal{H}$ are invariant subspaces for H_ω such that

$$H_\omega = \bigoplus_{j\in\mathbb{Z}} H_{\omega,j} = \bigoplus_{j\in\mathbb{Z}} \left(H_{0j} - \omega j\right).$$

In the momentum representation H_{0j} is a real multiplication operator and consequently $H_{\omega,j}$ with domain $\mathcal{D}_j = (H_{\omega,j} - i)^{-1}\mathcal{H}_j \subset \mathcal{H}_j$ is self-adjoint on $\mathcal{H}_j$. Let now

$$\mathcal{D}(H_\omega) := \left\{ f = \bigoplus_j f_j \;\middle|\; f_j \in \mathcal{D}_j,\ \sum_j \|H_{\omega,j}\, f_j\|_j^2 < \infty \right\}$$

with $\|\cdot\|_j$ being the norm in $\mathcal{H}_j$. The operator H_ω with the domain $\mathcal{D}(H_\omega)$ can be easily shown to be self-adjoint. Its domain is rotational invariant $R(t)\,\mathcal{D}(H_\omega) = \mathcal{D}(H_\omega)$ and the operator commutes with rotations.

The set $\mathcal{D}(H_\omega)$ is strictly larger than $\mathcal{D}(H_0) \cap \mathcal{D}(J)$. Indeed, consider a state $\Psi_0 \in \mathcal{H}$ with $\|\Psi_0\| = 1$ which in the momentum representation is given by the function $\widehat{\psi}_0 \in C_0^\infty$. We assume that

$$\operatorname{supp} \widehat{\psi}_0 \subset \{\mathbf{p} \in \mathbb{R}^\nu \mid |\mathbf{p}| < 1/2\}$$

and $\widehat{\psi}_0(\mathbf{p})$ is rotational symmetric such that $\int p_1\,|\widehat{\psi}_0(\mathbf{p})|^2\,\mathrm{d}p = 0$. For $n \in \mathbb{N}$ and $\omega \neq 0$ consider the sequence of normalized pairwise orthogonal vectors in $\mathcal{D}$ (3.4)

$$\widehat{\psi}_{\omega,n}(\mathbf{p}) := \exp\left\{in\frac{p_2}{2m\,\omega}\right\}\widehat{\psi}_0(\mathbf{p} - n\mathbf{e}_1),$$

Lemma 3.1. *Let* $F = \ x_1, x_2, \ldots, x_m \ $ *be of finite rank* m *and* R *normal in* F. *If* $s_1, s_2, \ldots, s_p$ *are non-identity elements of* R, $s_{p+1} \in R \cap \ {}_j(F)$ *with* $p + j = n$ $(n, j \geq 1)$ *and* *is any non-zero integer such that*

$$(s_1 - 1)(s_2 - 1) \cdots (s_p - 1)(s_{p+1} - 1) \in I^{n+1}(F),$$

then $s_{p+1} \in R \cap \ {}_{j+1}(F)$.

Proof. We may assume that F/R admits a pre-Abelian presentation where R is the normal closure $R = \ x_1^{\ 1} \ {}_1, x_2^{\ 2} \ {}_2, \ldots, x_m^{\ m} \ {}_m, \ {}_{m+1}, \ {}_{m+2}, \ldots \ ^F$ with ${}_m | \ {}_{m-1} | \cdots | \ {}_1 \geq 0$, ${}_i \in F'$, for $i = 1, 2, \ldots$ ([8], §3.3). Write $R = sgp\{r_1, r_2, \ldots, r_m, r_{m+1}, r_{m+2}, \ldots\}$, where $r_i = x_i^{\ i} \ {}_i$ for $1 \leq i \leq m$ and $r_i \in F'$ for $i \geq m+1$. We prove the result by induction on n. The case $n = 1$ follows easily. Assuming that the result holds for $n - 1$, we prove the result for n by induction on p. For $p = 0$, the result is a consequence of the facts that $(n+1)$st dimension subgroup of F is ${}_{n+1}(F)$ and $R \cap \ {}_n(F)/R \cap \ {}_{n+1}(F)$ is torsion-free. Let $s_1 \qquad {}_{i \geq 1} r_i^{m_i}$ (mod R'), then modulo $I^{n+1}(F)$,

$$\begin{aligned}
0 \quad & (s_1 - 1)(s_2 - 1) \cdots (s_p - 1)(s_{p+1} - 1) \\
& \sum_{i \geq 1} (r_i - 1)(s_2 - 1) \cdots (s_p - 1)(s_{p+1}^{m_i} - 1) \\
& \sum_{i=1}^{m} (x_i^{\ i} \ {}_i - 1)(s_2 - 1) \cdots (s_p - 1)(s_{p+1}^{m_i} - 1) \\
& \sum_{i=1}^{m} (x_i - 1)(s_2 - 1) \cdots (s_p - 1)(s_{p+1}^{m_i \ i} - 1).
\end{aligned} \tag{3.1}$$

Since $I(F)$ is a free right $\ F$-module with $\{x_i - 1 | 1 \leq i \leq m\}$ as a basis, it follows from (3.1) that

$$(s_2 - 1)(s_3 - 1) \cdots (s_p - 1)(s_{p+1}^{m_i \ i} - 1) \in I^n(F),$$

for all i, $1 \leq i \leq m$. The result now follows by induction hypotheses on n and p.

Theorem 3.2. *If* R, p *and* j *are as in Lemma* 3.1, *then*

$$\begin{aligned}
I^{n+1}(F) \cap I^p(R) I(R \cap \ {}_j(F)) &= I^{p+1}(R) I(R \cap \ {}_j(F)) \\
&\quad + I(R \cap F') I^{p-1}(R) I(R \cap \ {}_j(F)) + I^p(R) I(R \cap \ {}_{j+1}(F)).
\end{aligned}$$

Proof. We can suppose, by a standard reduction argument, that F is of finite rank m and F/R admits a pre-Abelian presentation as in Lemma 3.1. Let $\ \in I^{n+1}(F) \cap I^p(R) I(R \cap \ {}_j(F))$, then modulo $I^{p+1}(R) I(R \cap \ {}_j(F)) + I(R \cap F') I^{p-1}(R) I(R \cap \ {}_j(F))$,

$$\sum_{i \geq 1} (r_i - 1)(s_{2i} - 1) \cdots (s_{pi} - 1)(s_{(p+1)i} - 1),$$

$$(s_{ki} \in R, 2 \leq k \leq p \ \text{ and } \ s_{(p+1)i} \in R \cap \ {}_j(F))$$

$$\begin{aligned}
&\sum_{i=1}^{m}(x_i^{\ i}\ {}_i - 1)(s_{2i}-1)\cdots(s_{pi}-1)(s_{(p+1)i}-1) \qquad (3.2)\\
&\sum_{i=1}^{m}(x_i-1)(s_{2i}-1)\cdots(s_{pi}-1)(s_{(p+1)i}^{\ i}-1)\\
&\qquad\qquad (\mathrm{mod}\ \ I^2(F)I^{p-1}(R)I(R\cap\ \ {}_j(F)))\\
&0 \quad (\mathrm{mod}\ \ I^{n+1}(F)). \qquad (3.3)
\end{aligned}$$

$I(F)$ being free right F-module, it follows from (3.3) that

$$(s_{2i}-1)\cdots(s_{pi}-1)(s_{(p+1)i}^{\ i}-1)\in I^n(F),$$

for all i, $1\leq i\leq m$. Therefore, by Lemma 3.1, $s_{(p+1)i}\in R\cap\ \ {}_{j+1}(F)$ for all i and hence by (3.2),

$$\begin{aligned}
&\in I^{p+1}(R)I(R\cap\ \ {}_j(F))+I(R\cap F')I^{p-1}(R)I(R\cap\ \ {}_j(F))\\
&\quad + I^p(R)I(R\cap\ \ {}_{j+1}(F)).
\end{aligned}$$

The other way inclusion is straightforward.

Proof of Theorem 1.1. The proof follows by taking $j=1$ in Theorem 3.2 and by the fact that

$$\begin{aligned}
I(R\cap F')I^{n-1}(R) &\subset \sum_{i=1}^{n} I^{n-i}(R)I(\ \ {}_i(R)\cap\ \ {}_{i+1}(F))\\
&\subset I^{n+1}(F)\cap I^n(R).
\end{aligned}$$

Proof of Theorem 1.2. Splitting of the exact sequence (1.1) implies that the sequence

$$\begin{aligned}
&1\to G/HG'\quad H\cap G'/H'\to G/HG'\quad H/H'\\
&\quad\to G/HG'\quad H/H\cap G'\to 1
\end{aligned}$$

is split exact. Karan and Vermani [6] proved that

$$G/HG'\quad H/H'\cong I(G)I(H)/(I^2(G)I(H)+I^2(H)).$$

Using similar arguments, we can prove that

$$\begin{aligned}
G/HG'\quad H/H\cap G' \cong\ & I(G)I(H)/(I^2(G)I(H)\\
&+I(G)I(H\cap G')+I^2(H)).
\end{aligned}$$

Let ψ be the restriction of α (as defined in [6], Theorem 2.7) to $G/HG'\quad H\cap G'/H'$. Then ψ is one-one and

$$\mathrm{Im}\,\psi=(I^2(G)I(H)+I(G)I(H\cap G')+I^2(H))/(I^2(G)I(H)+I^2(H)).$$

Hence we have the commutative diagram

$$\begin{tikzcd}
\frac{G}{HG'} \quad \frac{H\cap G'}{H'} \arrow[r, tail] \arrow[d, "\cong"] & \frac{G}{HG'} \quad \frac{H}{H'} \arrow[r, two heads] \arrow[d, "\cong"] & \frac{G}{HG'} \quad \frac{H}{H\cap G'} \arrow[d, "\cong"] \\
\frac{I^2(G)I(H)+I(G)I(H\cap G')+I^2(H)}{I^2(G)I(H)+I^2(H)} \arrow[r, tail] & \frac{I(G)I(H)}{I^2(G)I(H)+I^2(H)} \arrow[r, two heads] & \frac{I(G)I(H)}{I^2(G)I(H)+I(G)I(H\cap G')+I^2(H)}.
\end{tikzcd}$$

Therefore

$$\begin{aligned}
\frac{G}{HG'} \quad \frac{H\cap G'}{H'} &\cong \frac{I^2(G)I(H)+I(G)I(H\cap G')+I^2(H)}{I^2(G)I(H)+I^2(H)} \\
&\cong \frac{I(G)I(H\cap G')}{(I^2(G)I(H)+I^2(H))\cap I(G)I(H\cap G')}.
\end{aligned}$$

Also

$$\frac{G}{HG'} \quad \frac{H\cap G'}{H'} \cong \frac{I(G)I(H\cap G')}{I^2(G)I(H\cap G')+I(G)I(H')+I(H)I(H\cap G')}$$

and we have the commutative diagram

$$\begin{tikzcd}
\frac{G}{HG'} \quad \frac{H\cap G'}{H'} \arrow[r, equal] \arrow[d, "\cong"] & \frac{G}{HG'} \quad \frac{H\cap G'}{H'} \arrow[d, "\cong"] \\
\frac{I(G)I(H\cap G')}{I^2(G)I(H\cap G')+I(G)I(H')+I(H)I(H\cap G')} \arrow[r, two heads, "\pi"] & \frac{I(G)I(H\cap G')}{(I^2(G)I(H)+I^2(H))\cap I(G)I(H\cap G')}
\end{tikzcd}$$

where π is the natural projection. An easy diagram chasing shows that π is a monomorphism. This completes the proof.

COROLLARY 3.3

If G and H are as in Theorem 1.2*, then*

$$\begin{aligned}
&(I^2(G)I(H)+I(G)I(H\cap G'))\cap I^2(H) \\
&\quad = I(H)I(H\cap G')+I^2(G)I(H)\cap I^2(H).
\end{aligned}$$

If R is any subgroup of F, then the exact sequence $1 \longrightarrow R\cap F'/R' \longrightarrow R/R' \longrightarrow R/R\cap F' \longrightarrow 1$ splits and the intersection $I^3(F)\cap I^2(R)$ can then be deduced from Corollary 3.3 by taking intersection via $I(F)I(R)$.

Theorem 3.4. *If R and S are subgroups of F, then*

$$\begin{aligned}
&FI(R)I(F)I(S)\cap I^2(R^F\cap S) \\
&\quad = I^3(R^F\cap S)+I(\ {}_2(R^F)\cap S)I(R^F\cap S)+I(R^F\cap S)I(R^F\cap S').
\end{aligned}$$

Proof. Observe that

$$
\begin{aligned}
&FI(R)I(F)I(S) \cap I^2(R^F \cap S)\\
&= \quad FI(R)I(F)I(S) \cap I(R^F \cap S)I(S) \cap I^2(R^F \cap S)\\
&= (\quad FI(R)I(F) \cap I(R^F \cap S) \quad S)I(S) \cap I^2(R^F \cap S),\\
&\qquad FI(S) \text{ being free left } \quad F\text{-module.}\\
&= (I(R^F \cap S)I(S) + \quad FI(R)I(F) \cap I(R^F \cap S))I(S) \cap I^2(R^F \cap S).
\end{aligned}
\tag{3.4}
$$

Let $x \in \quad FI(R)I(F) \cap I(R^F \cap S)$ and let $x = \sum_i m_i(a_i - 1)$, where $a_i \in R^F \cap S$. Take $h = \quad {}_i\, a_i^{m_i}$, then $h - 1 \quad x \pmod{I^2(R^F \cap S)}$. Thus $h - 1 \in \quad FI(R)I(F)$ and therefore $h \in \quad {}_2(R^F) \cap S$, by [5], Theorem 2.3. This, then implies that $x \in I^2(R^F \cap S) + I(\quad {}_2(R^F) \cap S)$ and therefore by (3.4)

$$
\begin{aligned}
&FI(R)I(F)I(S) \cap I^2(R^F \cap S)\\
&= (I(R^F \cap S)I^2(S) + I(\quad {}_2(R^F) \cap S)I(S)) \cap I^2(R^F \cap S)\\
&= I^3(R^F \cap S) + I(\quad {}_2(R^F) \cap S)I(R^F \cap S) + I(R^F \cap S)I(R^F \cap S'),
\end{aligned}
$$

by Theorem 2.3.

4. Identifications

We first prove the following:

Lemma 4.1. *If R and S are normal subgroups of F such that F/R and F/S are free-Abelian, then*

$$F \cap (1 + I^3(F) + I(F)I(R) + I(F)I(S)) = \quad {}_3(F)R'S'[R, S].$$

Proof. Observe that r.h.s. is contained in l.h.s.. For the reverse inequality we take $\quad \in F$ such that $\quad - 1 \in I^3(F) + I(F)I(R) + I(F)I(S)$ and move on to show that 1 (mod $\quad {}_3(F)R'S'[R, S]$). Since F/F' is free-Abelian, it follows that each of R/F' and S/F' is a direct summand of F/F'. We first choose a basis $x_1, x_2, \ldots, x_1', x_2', \ldots$ for F such that $x_1', x_2', \ldots$ is a basis for R (modulo F'). By [9, Theorem 3.11], $\quad \in [R, F][S, F]$ and modulo $[S, F]$,

$$\quad [x_i', f_i], \quad {}_{i \geq 1} \tag{4.1}$$

where $f_i \in F$ and therefore modulo $I^3(F) + I(F)I(R) + I(F)I(S)$,

$$
\begin{aligned}
0 \qquad & - 1\\
& \sum_{i \geq 1}((x_i' - 1)(f_i - 1) - (f_i - 1)(x_i' - 1))\\
& \sum_{i \geq 1}(x_i' - 1)(f_i - 1).
\end{aligned}
\tag{4.2}
$$

Therefore if $\ \in F \cap (1 + I(R)I^3(F)) \subset F \cap (1 + I(R)I(F)) = R'$ ([5, Proposition 2.1]), then by (4.6)

$$\ \in R \cap (1 + I^4(R) + I^2(R)I(R \cap F') + I(R)I(R \cap \ _3(F))).$$

Since $R/R \cap F'$ is free-Abelian, using Theorem 2.2 and the filtration map ϕ, we make belong to

$$(R \cap F') \cap (1 + I^3(R \cap F') + I(R')I(R \cap F') \\ + I(R \cap F')I(R \cap \ _3(F))) \ _4(R).$$

Now using similar arguments as in the proof of Theorem 1.3, we can show that

$$\in \ _4(R) \ _2(R \cap \ _3(F))[R \cap F', R' \cap \ _3(F)].$$

The reverse inclusion being trivial, we have

$$F \cap (1 + I(R)I^3(F)) = \ _4(R) \ _2(R \cap \ _3(F))[R \cap F', R' \cap \ _3(F)].$$

Acknowledgement

This research was supported by CSIR, New Delhi.

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